**Exploring Logarithmic Differentiation Through Examples**

**Abstract**

In this exploration, I delved into the technique of logarithmic differentiation, showcasing its utility in handling complex functions where both the base and exponent are variable. Through detailed examples, I demonstrated how this method simplifies derivative computation for intricate expressions, including products and quotients involving chain rules. The process leverages logarithmic properties to transform equations into more manageable forms, reducing computational complexity and enhancing accuracy.

When dealing with functions that defy traditional power or exponential rules due to variable bases and exponents, I often turn to **logarithmic differentiation**. This approach provides a systematic way to simplify and compute derivatives for these complex functions.

**Why Logarithmic Differentiation?**

Logarithmic differentiation becomes necessary when functions like y=xxy = \sqrt{x^x}y=xx​ are encountered. This function combines a variable base (xxx) and a variable exponent (xxx), making it neither a straightforward power function (xnx^nxn) nor a simple exponential function (axa^xax). Traditional differentiation rules struggle here, but logarithmic differentiation simplifies the process.

**Example 1: y=xxy = \sqrt{x^x}y=xx​**

To start, I rewrote the function:

y=xx/2y = x^{x/2}y=xx/2

Taking the natural log of both sides transformed it:

ln⁡(y)=x2ln⁡(x)\ln(y) = \frac{x}{2} \ln(x)ln(y)=2x​ln(x)

This setup allowed me to apply implicit differentiation:

1ydydx=12ln⁡(x)+x2⋅1x\frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \ln(x) + \frac{x}{2} \cdot \frac{1}{x}y1​dxdy​=21​ln(x)+2x​⋅x1​

Simplifying further:

dydx=(12ln⁡(x)+12)y\frac{dy}{dx} = \left(\frac{1}{2} \ln(x) + \frac{1}{2}\right) ydxdy​=(21​ln(x)+21​)y

Finally, substituting back y=xx/2y = x^{x/2}y=xx/2, I arrived at:

dydx=(12ln⁡(x)+12)xx/2\frac{dy}{dx} = \left(\frac{1}{2} \ln(x) + \frac{1}{2}\right) x^{x/2}dxdy​=(21​ln(x)+21​)xx/2

This method streamlined the process and avoided cumbersome chain rules.

**Example 2: y=(2x+1)3×(4−x2)5y = (2x + 1)^3 \times (4 - x^2)^5y=(2x+1)3×(4−x2)5**

For this more intricate product, logarithmic differentiation proved invaluable. Taking the natural log of both sides:

ln⁡(y)=3ln⁡(2x+1)+5ln⁡(4−x2)\ln(y) = 3\ln(2x + 1) + 5\ln(4 - x^2)ln(y)=3ln(2x+1)+5ln(4−x2)

Differentiating implicitly:

1ydydx=32x+1⋅2−54−x2⋅2x\frac{1}{y} \frac{dy}{dx} = \frac{3}{2x + 1} \cdot 2 - \frac{5}{4 - x^2} \cdot 2xy1​dxdy​=2x+13​⋅2−4−x25​⋅2x

Simplifying:

dydx=y(62x+1−10x4−x2)\frac{dy}{dx} = y \left(\frac{6}{2x + 1} - \frac{10x}{4 - x^2}\right)dxdy​=y(2x+16​−4−x210x​)

Substituting back y=(2x+1)3×(4−x2)5y = (2x + 1)^3 \times (4 - x^2)^5y=(2x+1)3×(4−x2)5:

dydx=(2x+1)3(4−x2)5(62x+1−10x4−x2)\frac{dy}{dx} = (2x + 1)^3 (4 - x^2)^5 \left(\frac{6}{2x + 1} - \frac{10x}{4 - x^2}\right)dxdy​=(2x+1)3(4−x2)5(2x+16​−4−x210x​)

This method sidestepped the need for applying the product and chain rules repeatedly, saving significant time and effort.

**Example 3: Simplifying Quotients**

For y=2x−1x+2y = \frac{\sqrt{2x - 1}}{x + 2}y=x+22x−1​​, logarithmic differentiation efficiently handled the square root and quotient:

ln⁡(y)=12ln⁡(2x−1)−ln⁡(x+2)\ln(y) = \frac{1}{2} \ln(2x - 1) - \ln(x + 2)ln(y)=21​ln(2x−1)−ln(x+2)

Differentiating:

1ydydx=12⋅12x−1⋅2−1x+2\frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{2x - 1} \cdot 2 - \frac{1}{x + 2}y1​dxdy​=21​⋅2x−11​⋅2−x+21​

Simplifying and substituting yyy:

dydx=2x−1x+2(12x−1−1x+2)\frac{dy}{dx} = \frac{\sqrt{2x - 1}}{x + 2} \left(\frac{1}{2x - 1} - \frac{1}{x + 2}\right)dxdy​=x+22x−1​​(2x−11​−x+21​)

**Observations**

Logarithmic differentiation shines when traditional rules falter due to the complexity of the functions. It simplifies expressions by leveraging logarithmic properties, allowing me to focus on efficient computation rather than wrestling with nested rules. With practice, this method becomes an indispensable tool for handling challenging derivatives in calculus.